

SUPERSTRINGS IN ADS IN LIGHT CONE GAUGE

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We discuss light-cone gauge description of type IIB Green-Schwarz superstring in $AdS_5 \times S^5$ with a hope to make progress towards understanding spectrum of this theory. As in flat space, fixing light-cone gauge consists of two steps: (i) fixing kappa symmetry in such a way that the fermionic part of the action does not depend on x^- ; (ii) fixing 2-d reparametrizations by $x^+ = \tau$ and a condition on 2-d metric. In curved AdS space the latter cannot be the standard conformal gauge and breaks manifest 2-d Lorentz invariance. It is natural, therefore, to work in phase-space framework, imposing the GGRT light-cone gauge conditions $x^+ = \tau$, $P^+ = \text{const}$. We obtain the resulting light-cone superstring Hamiltonian.

1. Introduction

One lesson of developments in the last 4 years is that string theory duals of $(\mathcal{N} = 4, 2, 1, 0)$ supersymmetric gauge theories should be fermionic strings in 4+1+more dimensions propagating in curved space with Ramond-Ramond backgrounds (see 1,2 and, e.g., 3,4,5 for reviews and references). It is important to make progress in understanding properties of such string theories.

The basic most symmetric example is (large N) $\mathcal{N}=4$ Super Yang-Mills theory dual to (weakly coupled) type IIB superstring theory in $AdS_5 \times S^5$ space with Ramond-Ramond flux ². Though this background has a lot of symmetry, solving the corresponding string theory appears to be a complicated problem. The commonly used procedure, in the known exactly solvable cases, is to start with a string action, solve the classical string equations, then quantize the theory, find the string spectrum, vertex operators, scattering amplitudes, etc. Even the first steps in this program are nontrivial in the $AdS_5 \times S^5$ case.

In contrast to other known cases, here fermionic degrees of freedom play crucial role and cannot be ignored from the start. The bosonic part of the string sigma model is the sum of the two symmetric space chiral models – $SO(2,4)/SO(1,5)$ and $SO(6)/SO(5)$. These are not conformally invariant ($R_{mn} \neq 0$), and it is the fermionic R-R couplings ($\bar{\theta}\gamma^{\dots}\theta\partial x\partial x F_5\dots$) that “glue” the AdS_5 and S^5 bosons together ensuring the vanishing of the 2-d beta-function ($R_{mn} - (F_5^2)_{mn} = 0$).

The presence of a curved R-R background indicates that one should use the manifestly supersymmetric Green-Schwarz ⁶ description of superstring. Finding an

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explicit expression for curved space GS action $S(x, \theta)$ is difficult in general (one needs to know the component expansion of the background supergravity superfields which enter the formal expression ⁷ for the action). In the present $AdS_5 \times S^5$ case, this technical problem has a nice geometrical solution ⁸ based on viewing string as moving on the supercoset $PSU(2, 2|4)/[SO(1, 4) \times SO(5)]$ which replaces the flat superspace (10-d super Poincare)/ $SO(1, 9)$ in original GS construction.

The resulting action, though explicitly known ^{8,9,10}, looks highly nonlinear containing terms of many orders in θ . This is, however, an illusion of complexity: the fermionic part of the action simplifies dramatically in proper κ -symmetry gauges – it becomes quadratic and quartic in θ only ^{11,12,13}. Its structure is similar to that of the flat space GS action in a covariant κ -symmetry gauge which is also quartic in fermions.^a

If one interprets the $AdS_5 \times S^5$ supergroup $PSU(2, 2|4)$ as the $\mathcal{N} = 4$ superconformal group in 4 dimensions, it is natural to split the fermionic generators into 4 standard supergenerators Q_i and 4 special conformal supergenerators S_i (we suppress the 4-d spinor indices). The associated superstring coordinates will be denoted as θ_i and η_i (we use fermionic parametrization of ^{14,15}). The 4-d Lorentz covariant κ -symmetry “S-gauge” ¹⁴ (which leads to an action equivalent to the one in ^{11,12}) corresponds to setting all η_i to zero. The resulting superstring Lagrangian written in “4+6” parametrization of $AdS_5 \times S^5$ where ($R = 1$)

$$ds^2 = Y^2 dx^a dx^a + Y^{-2} dY^M dY^M, \quad (1)$$

or, equivalently, ($a = 0, \dots, 3$, $M = 1, \dots, 6$, $Y^M = e^\phi u^M$)

$$ds^2 = e^{2\phi} dx^a dx^a + d\phi^2 + du^M du^M, \quad u^M u^M = 1, \quad (2)$$

has the following simple structure ^{14b}

$$\begin{aligned} L = & -\frac{1}{2} \sqrt{g} \left(Y^2 [\partial_\mu x^a - (i\theta_{a\dot{a}} \sigma^{a\dot{a}b} \partial_\mu \theta_{\dot{b}}^i + h.c.)]^2 + Y^{-2} \partial^\mu Y^M \partial_\mu Y^M \right) \\ & - (i\epsilon^{\mu\nu} \partial_\mu Y^M \theta^{\dot{a}i} \rho_{ij}^M \partial_\nu \theta_{\dot{a}}^j + h.c.) . \end{aligned} \quad (3)$$

This covariant κ -symmetry gauge fixed action is well-defined and useful for developing perturbation theory near classical “long” string configurations ending at the boundary of AdS_5 ^{13,16,17}, e.g., the ones appearing in Wilson loop computations ¹⁸. However, the kinetic term of the fermions $\sim \partial x \theta \partial \theta$ is degenerate for “short” strings ¹³, and thus this action is not directly applicable for computing the spectrum of “short” closed string in the bulk of $AdS_5 \times S^5$.

^aFor example, in $\theta^1 = \theta^2$ gauge the flat space type IIB GS action has the following structure: $S \sim \int [(\partial x)^2 + \partial x \bar{\theta} \gamma \partial \theta + (\bar{\theta} \partial \theta)^2]$.

^bThe actions in ^{11,12} have isomorphic form, corresponding to a specific choice of the 10-d Dirac matrix representation. We use “chiral” representations for the 4-d and 6-d Dirac matrices, $\gamma^a = \begin{pmatrix} 0 & \sigma^a \\ \bar{\sigma}^a & 0 \end{pmatrix}$, $\gamma^M = \begin{pmatrix} 0 & \rho^{Mij} \\ \rho_{ij}^M & 0 \end{pmatrix}$, with $(\rho^M)^{ij} \equiv -(\rho_{ij}^M)^*$, $i, j = 1, 2, 3, 4$. $\mathbf{a}, \mathbf{b} = 1, 2$ are the $sl(2, C)$ (4-d spinor) indices and $\theta_{\mathbf{a}\dot{i}}^\dagger = -\theta_{\dot{i}}^{\mathbf{a}}$, $\theta_{\dot{i}}^{\mathbf{a}\dagger} = \theta^{\mathbf{a}\dot{i}}$. The 10-d spinors are split as (θ^i, θ_i) .

In order to avoid this degeneracy problem, it is natural to try to follow the same approach which worked remarkably well in flat space^{19,6}: use light-cone gauge (for an alternative covariant approach to quantization of GS action see²⁰). In flat space superstring light-cone gauge fixing consists of the two steps: (i) fermionic light-cone gauge choice, i.e. fixing the κ -symmetry by $\gamma^+\theta^I = 0$; (ii) bosonic light-cone gauge choice, i.e. using the conformal gauge^c $\sqrt{g}g^{\mu\nu} = \eta^{\mu\nu}$ and fixing the residual conformal diffeomorphism symmetry by $x^+(\tau, \sigma) = p^+\tau$. Fixing the fermionic light-cone gauge already produces a substantial simplification of the flat-space GS action: it becomes quadratic in θ . Similarly, in $AdS_5 \times S^5$ the first task should be to find a light-cone κ -symmetry gauge in which the fermion kinetic term becomes $\partial x^+ \theta \partial \theta$, i.e. involves only one combination – x^+ – of 4-d coordinates, so that the non-degeneracy of the kinetic term for the fermions will not depend on a choice of a specific string background in transverse directions. To simplify the fermion kinetic term further one should then try to choose the light-cone bosonic gauge $x^+ = \tau$.

Fixing the fermionic light-cone gauge was discussed in detail in^{14,d} and the fixing the bosonic part of light-cone gauge and derivation of the resulting light-cone Hamiltonian was described in¹⁵. Our aim below is to review these results.

The metric (2) corresponds to the AdS space in the Poincaré coordinate patch. One needs to use the Poincaré coordinates to have a null isometry in the bulk and at the boundary (the boundary should have $R^{1,3}$ topology). The AdS/CFT duality suggests that since the boundary SYM theory in $R^{1,3}$ has a well-defined light-cone gauge description²², it should be possible to fix some analog of a light-cone gauge for the dual string theory as well.

The SYM theory does not admit a manifestly $\mathcal{N} = 4$ supersymmetric Lorentz-covariant description, but has a simple superspace description in the light-cone gauge $A^+ = 0$ ²². It is based on a single chiral superfield $\Phi(x, \theta) = A(x) + \theta^i \psi_i(x) + \dots$ where $A = A_1 + iA_2$ represents the transverse components of the gauge field and ψ_i its fermionic partner which transforms under the fundamental representation of R-symmetry group $SU(4)$. In addition to the standard light-cone supersymmetry (shifts of θ), the light-cone superspace SYM action $S[\Phi]$ has also a non-linear superconformal symmetry. This suggests that it may be possible to formulate the bulk string theory in a way which is naturally related to the light-cone form of the boundary SYM theory. In particular, it may be useful to split the fermionic string coordinates into the two parts with manifest $SU(4) \simeq SO(6)$ transformation properties which will be the counterparts of the linearly realized Poincaré supersymmetry supercharges Q_i and the nonlinearly realized conformal supersymmetry supercharges S_i of the SYM theory.

As was shown in^{23,24,25}, field theories in AdS space (in particular, type IIB supergravity) admit a simple light-cone description. There exists a light-cone Hamiltonian for a superparticle in $AdS_5 \times S^5$ which may be used to formulate AdS/CFT correspondence between chiral primary states of SYM theory and supergravity states

^cWe use Minkowski signature 2-d world sheet metric $g_{\mu\nu}$ with $g \equiv -\det g_{\mu\nu}$.

^dSome previous work in this direction using different approach was reported in²¹.

of the bulk theory in the light-cone gauge. This indicates that the full superstring theory in $AdS_5 \times S^5$ should also have a kind of light-cone gauge formulation, which may be useful in the context of the AdS/CFT correspondence. The string light-cone Hamiltonian should reduce to the flat space one in the $R \rightarrow \infty$ limit and to the superparticle Hamiltonian²⁵ in the zero slope limit.

This is the motivation behind our light-cone gauge fixing program^{14,15}, in spite of the fact that there is no globally well-defined null Killing vector in AdS space (its norm proportional to $e^{2\phi}$ vanishes at the horizon $\phi = -\infty$ of the Poincaré patch). We use a formal approach, assuming that the degeneracy of the light-cone description reflected in the $e^{2\phi} \rightarrow 0$ singularity of the resulting light-cone gauge fixed action should have some physical resolution, e.g., the form of the light-cone Hamiltonian may suggest how the wave functions should be defined in this region.

2. Superstring Action in light-cone kappa symmetry gauge

In¹⁴ it was shown how the light-cone type κ -symmetry gauge can be fixed in the original action of⁸ so that the resulting action has indeed the required form: only ∂x^+ appears in the fermionic kinetic term. In contrast to the Lorentz covariant “S-gauge” where 16 fermions η_i are set equal to zero, the light-cone gauge used in¹⁴ corresponds to setting to zero “half” of the 16 θ_i and “half” of the 16 η_i (“half” is defined with respect to $SO(1,1)$ rotations in the light-cone directions). The remaining fermions (which will be again denoted by θ_i, η_i) from now on are simply 4+4 complex anticommuting variables carrying no extra Lorentz spinor indices. For comparison, the flat space GS action in the light-cone gauge ($\gamma^+\theta = 0$)¹⁹, written in a similar parametrization of the 16 fermionic coordinates, has the following structure:

$$\mathcal{L} = -\frac{1}{2}\sqrt{g}(\partial_\mu x)^2 - \left[\frac{i}{2}\sqrt{g}\partial^\mu x^+(\theta\partial_\mu\theta + \eta\partial_\mu\eta) - \epsilon^{\mu\nu}\partial_\mu x^+\eta\partial_\nu\theta + h.c. \right]. \quad (4)$$

Note that θ ’s and η ’s enter diagonally in the kinetic term, but they are mixed in the WZ term. This form of the original GS Lagrangian is the flat space limit of the light-cone $AdS_5 \times S^5$ Lagrangian of¹⁴ (cf. (3))

$$\begin{aligned} \mathcal{L} = & -\sqrt{g}\left[Y^2(\partial^\mu x^+\partial_\mu x^- + \partial^\mu x\partial_\mu \bar{x}) + \frac{1}{2}Y^{-2}(\partial_\mu Y^M + i\eta\rho^{MN}\eta Y^N Y^2\partial_\mu x^+)^2\right] \\ & - \frac{i}{2}\sqrt{g}g^{\mu\nu}Y^2\partial_\mu x^+\left[\theta^i\partial_\nu\theta_i + \theta_i\partial_\nu\theta^i + \eta^i\partial_\nu\eta_i + \eta_i\partial_\nu\eta^i + iY^2\partial_\nu x^+(\eta^2)^2\right] \\ & + \epsilon^{\mu\nu}|Y|\partial_\mu x^+\left[\eta^i\rho_{ij}^MY^M(\partial_\nu\theta^j - i\sqrt{2}|Y|\eta^j\partial_\nu x) + h.c.\right]. \end{aligned} \quad (5)$$

We decomposed x^a into the light-cone and 2 complex coordinates: (x^+, x^-, x, \bar{x}) , $x^\pm = \frac{1}{\sqrt{2}}(x^3 \pm x^0)$; $x, \bar{x} = \frac{1}{\sqrt{2}}(x^1 \pm ix^2)$.^e Written in terms of the radial direction ϕ

^eWe use the following notation: 4-d indices are $a, b = 0, 1, 2, 3$; $SO(6)$ indices are $M, N = 1, \dots, 6$; $SU(4)$ indices are $i, j = 1, 2, 3, 4$; 2-d indices are $\mu, \nu = 0$. The matrices ρ^M (off-diagonal blocks of 6-d Dirac matrices in chiral representation) satisfy: $\rho_{ij}^M = -\rho_{ji}^M$, $(\rho^M)^{il}\rho_{lj}^N + (\rho^N)^{il}\rho_{lj}^M = 2\delta^{MN}\delta_j^i$, $(\rho^M)^{ij} \equiv -(\rho_{ij}^M)^*$, and $\rho^{MN} \equiv \rho^{[M}\rho^{\dagger N]}$, i.e. $\rho_j^{MNi} = \frac{1}{2}(\rho^M)^{il}\rho_{lj}^N - (M \leftrightarrow N)$. Also, $\theta_i^\dagger = \theta^i$, $\eta_i^\dagger = \eta^i$, $\theta^2 \equiv \theta^i\theta_i$, $\eta^2 \equiv \eta^i\eta_i$.

of AdS_5 and unit 6-d vector u^M parametrizing S^5 this Lagrangian becomes ($Y^M = e^\phi u^M$, $|Y| = e^\phi$)¹⁴

$$\begin{aligned} \mathcal{L} = & -\sqrt{g} \left[e^{2\phi} (\partial^\mu x^+ \partial_\mu x^- + \partial^\mu x \partial_\mu \bar{x}) + \frac{1}{2} \partial^\mu \phi \partial_\mu \phi + \frac{1}{2} (\partial_\mu u^M + i \eta \rho^{MN} \eta u^N e^{2\phi} \partial_\mu x^+)^2 \right] \\ & - \frac{i}{2} \sqrt{g} g^{\mu\nu} e^{2\phi} \partial_\mu x^+ \left[\theta^i \partial_\nu \theta_i + \theta_i \partial_\nu \theta^i + \eta^i \partial_\nu \eta_i + \eta_i \partial_\nu \eta^i + i e^{2\phi} \partial_\nu x^+ (\eta^2)^2 \right] \\ & + \epsilon^{\mu\nu} e^{2\phi} \partial_\mu x^+ \left[\eta^i \rho_{ij}^M u^M (\partial_\nu \theta^j - i \sqrt{2} e^\phi \eta^j \partial_\nu x) + h.c. \right]. \end{aligned} \quad (6)$$

This action has several important properties:

(a) It contains x^- only in the bosonic part and only linearly (in $\partial x^+ \partial x^-$ term), and the fermion kinetic terms are multiplied by the derivative of x^+ only. It is the crucial property of this light-cone κ -symmetry gauge fixed form of the action that the fermion kinetic term involves the derivative of only *one* space-time direction – x^+ , i.e. its (non)degeneracy does not depend on transverse string profile. One expects, therefore, that after one fixes the bosonic light-cone gauge, it should this action should be a well defined starting point for quantizing the theory in the “short” string sector.

(b) The fermionic κ -symmetry light-cone gauge we used reduces the 32 fermionic coordinates θ_α^I (two left Majorana-Weyl 10-d spinors) to 16 physical Grassmann variables: “linear” θ^i and “nonlinear” η^i and their hermitian conjugates θ_i and η_i , which transform in fundamental representations of $SU(4)$. The superconformal algebra $psu(2, 2|4)$ implies that these are related to the Poincaré and the conformal supersymmetry in the light-cone gauge description of the boundary theory. The action and symmetry generators have simple (quadratic) dependence on θ^i , but complicated (quartic) dependence on η^i . The action is symmetric under shifting $\theta \rightarrow \theta + \epsilon$ (supplemented by a shift of x^-). It is this symmetry that is responsible for the fact that the theory is linear in θ , i.e. that there is no quartic terms in θ .

(c) The fact that the action has only quadratic and quartic fermionic terms has to do with special symmetries of the $AdS_5 \times S^5$ background (covariantly constant curvature and 5-form field strength). The presence of the η^4 term reflects the curvature of the background.^f As follows from the discussion in⁸, the ‘extra’ $O(\eta^2)$ terms should have the interpretation of the couplings to the R-R 5-form background.

(d) The $AdS_5 \times S^5$ superstring action in general depends on two parameters: the scale (equal radii) R of $AdS_5 \times S^5$ and the inverse string tension or α' . Restoring the dependence on R one finds that in the flat space limit $R \rightarrow \infty$ the quartic fermionic term goes away, while the kinetic term reduces to the flat space light-cone GS action after representing each of the two $SO(8)$ spinors in terms of the two $SU(4)$ spinors.

^fNote that the light-cone gauge GS action in a curved space of the form $R^{1,1} \times M^8$ with generic NS-NS and R-R backgrounds²⁶ (reconstructed from the light-cone flat space GS vertex operators) contains, in general, higher than quartic fermionic terms, multiplied by higher derivatives of the background fields. This light-cone GS action has quartic fermionic term^{27,26} involving the curvature tensor $R_{\dots} \partial x^+ \partial x^+ (\bar{\theta} \gamma^{\dots} \theta) (\bar{\theta} \gamma^{\dots} \theta) \sim R_{\dots} (p^+)^2 (\bar{\theta} \gamma^{\dots} \theta) (\bar{\theta} \gamma^{\dots} \theta)$ which is similar to the one present in the NSR string action (i.e. in the standard 2-d supersymmetric sigma model).

3. Fixing the bosonic light-cone gauge

As a next step to quantization of the theory one would like, as in the flat case, to eliminate the ∂x^+ -factors from the fermion kinetic terms in (6). In flat space this was possible by choosing the bosonic light-cone gauge. In the Polyakov formulation this may be done by fixing the conformal gauge $\sqrt{g}g^{\mu\nu} = \eta^{\mu\nu}$, and then noting that since the resulting equation $\partial^2 x^+ = 0$ has the general solution $x^+(\tau, \sigma) = f(\tau - \sigma) + h(\tau + \sigma)$, one can fix the residual conformal diffeomorphism symmetry on the plane by choosing $x^+(\tau, \sigma) = p^+ \tau$.

Let us first not make any explicit gauge choice and consider the superstring path integral assuming that there is no sources for x^- . The linear dependence of the action (5),(6) on x^- allows us to integrate over x^- explicitly: we get δ -function constraint imposing the equation of motion $\partial_\mu(\sqrt{g}g^{\mu\nu}e^{2\phi}\partial_\nu x^+) = 0$ for x^+ , which is formally solved by setting

$$\sqrt{g}g^{\mu\nu}e^{2\phi}\partial_\nu x^+ = \epsilon^{\mu\nu}\partial_\nu f, \quad \text{i.e.} \quad e^{2\phi}\partial_\mu x^+ = g_{\mu\nu}\frac{\epsilon^{\nu\lambda}}{\sqrt{g}}\partial_\lambda f. \quad (7)$$

where $f(\tau, \sigma)$ is an arbitrary function. Since our action (6) depends on x^+ only through $e^{2\phi}\partial x^+$, we are then able to integrate over x^+ as well, eliminating it in favor of the function f . The resulting fermion kinetic term is then non-degenerate (for a properly chosen f), and may be interpreted as an action of 2-d fermions in curved 2-d geometry determined by f and $g_{\mu\nu}$ (cf. ^{17,14} and refs. there).

To proceed further, one would like to fix 2-d diffeomorphisms by a condition on $g_{\mu\nu}$ and a condition on f corresponding to the usual light-cone gauge $x^+ = \tau$. An important observation ^{28,29,14} is that in the case of the *AdS* type curved spaces, the bosonic light-cone gauge $x^+ = \tau$ can not be combined with the standard conformal gauge $\sqrt{g}g^{\mu\nu} = \eta^{\mu\nu}$ (imposing the conformal gauge one in general is unable to solve the equation for x^+ by $x^+ = \tau$). Instead, one needs to impose a condition on $g_{\mu\nu}$ that breaks the manifest 2-d Lorentz symmetry and leads to a rather non-standard string action, with all terms coupled to the radial function ϕ of *AdS*₅ space.

A consistent gauge choice is ¹⁴

$$e^{2\phi}\sqrt{g}g^{00} = -1, \quad x^+ = p^+ \tau. \quad (8)$$

Indeed, the equation for x^+ is then satisfied identically. This choice is equivalent to $f = \sigma$ in (7). A closely related alternative, originally suggested by Polyakov ²⁹, is a modification of the conformal gauge $\sqrt{g}g^{ab} = \text{diag}(-e^{-2\phi}, e^{2\phi})$ which is also consistent with the light-cone gauge condition $x^+ = \tau$.

The resulting action has *AdS*₅ radial direction ϕ factors coupled differently to ∂_0 and ∂_1 derivative terms, and the S^5 part of the action is also coupled to ϕ . The absence of manifest 2-d Lorentz symmetry suggests that it is natural to use the *phase space* formulation of the light-cone gauge fixed theory. The coordinate space Polyakov approach based on (8) is equivalent to the phase space approach based on fixing the diffeomorphisms by $x^+ = p^+ \tau$, $P^+ = \text{const}$ ¹⁵. In general, the original

GGRT phase space approach³⁰ to light-cone gauge fixing is directly applicable in the present curved space case.

To illustrate the derivation of phase space Lagrangian let us consider first the example of a classical bosonic string in AdS_5 ($2\pi\alpha' = 1$, $R = 1$)

$$\mathcal{L} = -h^{\mu\nu} \left(\partial_\mu x^+ \partial_\nu x^- + \partial_\mu x \partial_\nu \bar{x} + \frac{1}{2} e^{-2\phi} \partial_\mu \phi \partial_\nu \phi \right) , \quad h^{\mu\nu} \equiv \sqrt{g} g^{\mu\nu} e^{2\phi} . \quad (9)$$

Introducing the momenta \mathcal{P}^a for the light-cone coordinates (x^+, x^-) and the two transverse coordinates (x, \bar{x}) and the momentum π for the radial direction ϕ we get

$$\begin{aligned} \mathcal{L} = & \dot{x}_\perp \mathcal{P}_\perp + \dot{\phi} \pi + \dot{x}^+ \mathcal{P}^- + \dot{x}^- \mathcal{P}^+ \\ & + \frac{1}{2h^{00}} \left[(\mathcal{P}_\perp^2 + 2\mathcal{P}^+ \mathcal{P}^-) + e^{4\phi} (\dot{x}_\perp^2 + 2\dot{x}^+ \dot{x}^-) + e^{2\phi} (\pi^2 + \dot{\phi}^2) \right] \\ & + \frac{h^{01}}{h^{00}} (\dot{x}_\perp \mathcal{P}_\perp + \dot{\phi} \pi + \dot{x}^+ \mathcal{P}^- + \dot{x}^- \mathcal{P}^+) , \end{aligned} \quad (10)$$

where $1/h^{00}$ and h^{01}/h^{00} play the role of the Lagrange multipliers imposing two constraints. Choosing the light-cone gauge $x^+ = \tau$, $\mathcal{P}^+ = p^+ = \text{const}$ and integrating over \mathcal{P}^- we get the relation

$$h^{00} = -p^+ , \quad (11)$$

which is equivalent (up to a rescaling) to the condition on the metric in (8). The expression for \mathcal{P}^- follows from the $1/h^{00}$ constraint after using the h^{01}/h^{00} constraint. The resulting light-cone gauge Hamiltonian density is¹⁵

$$\mathcal{H} = -\mathcal{P}^- = \frac{1}{2p^+} \left[\mathcal{P}_\perp^2 + e^{4\phi} \dot{x}_\perp^2 + e^{2\phi} (\pi^2 + \dot{\phi}^2) \right] . \quad (12)$$

As usual, the coordinate x^- does not appear in the Hamiltonian, but is determined from the reparametrization constraint

$$p^+ \dot{x}^- + \mathcal{P}_\perp \dot{x}_\perp + \dot{\phi} \pi = 0 . \quad (13)$$

For closed string, the integral of this constraint over σ constrains the state space to the subspace invariant under σ translations.

For comparison, for string in AdS_3 described by $SL(2, R)$ WZW model (in the standard Gauss parametrization)

$$\mathcal{L} = -\sqrt{g} g^{\mu\nu} (e^{2\phi} \partial_\mu x^+ \partial_\nu x^- + \frac{1}{2} \partial_\mu \phi \partial_\nu \phi) + \epsilon^{\mu\nu} e^{2\phi} \partial_\mu x^+ \partial_\nu x^- , \quad (14)$$

one finds

$$\mathcal{H} = \frac{e^{2\phi}}{2p^+} (\pi + \dot{\phi})^2 , \quad p^+ \dot{x}^- + \pi \dot{\phi} = 0 . \quad (15)$$

4. Light cone superstring Hamiltonian

Repeating the same procedure for the superstring action in (6) we get ¹⁵ g

$$\begin{aligned} \mathcal{L} = & \mathcal{P}_\perp \dot{x}_\perp + \pi \dot{\phi} + \mathcal{P}_M \dot{u}^M + \frac{i}{2} p^+ (\theta^i \dot{\theta}_i + \eta^i \dot{\eta}_i + \theta_i \dot{\theta}^i + \eta_i \dot{\eta}^i) + \mathcal{P}^- \\ & - \frac{h^{01}}{p^+} \left[p^+ \dot{x}^- + \mathcal{P}_\perp \dot{x}_\perp + \pi \dot{\phi} + \mathcal{P}_M \dot{u}^M + \frac{i}{2} p^+ (\theta^i \dot{\theta}_i + \eta^i \dot{\eta}_i + \theta_i \dot{\theta}^i + \eta_i \dot{\eta}^i) \right], \end{aligned} \quad (16)$$

where the light-cone Hamiltonian density is ($\mathcal{H} = -\mathcal{P}^-$)

$$\begin{aligned} \mathcal{H} = & \frac{1}{2p^+} \left(\mathcal{P}_\perp^2 + e^{4\phi} \dot{x}_\perp^2 \right. \\ & + e^{2\phi} [\pi^2 + \dot{\phi}^2 + (l^i_j)^2 + \dot{u}^M \dot{u}^M + p^{+2} (\eta^2)^2 + 4p^+ \eta_i l^i_j \eta^j] \\ & \left. - e^{2\phi} u^M \left[\eta^i \rho^M_{ij} (\dot{\theta}^j - i\sqrt{2} e^\phi \eta^j \dot{x}) + \eta_i \rho^{Mij} (\dot{\theta}_j + i\sqrt{2} e^\phi \eta_j \dot{x}) \right] \right), \end{aligned} \quad (17)$$

and h^{01} imposes the reparametrization constraint. Here \mathcal{P}^M is the momentum corresponding to the unit 6-vector u^M , $u^M \mathcal{P}^M = 0$, and $l^i_j \equiv \frac{i}{2} (\rho^{MN})^i_j u^M \mathcal{P}^N$, so that $(l^i_j)^2 \equiv l^i_j l^j_i = \mathcal{P}^M \mathcal{P}^M$. In the particle theory limit l^i_j has the interpretation of the $SU(4)$ angular momentum operator ²⁵ and (17) reduces to the (classical limit of) light-cone Hamiltonian for a superparticle in $AdS_5 \times S^5$ space ²⁵

$$\mathcal{H} = \frac{1}{2p^+} \left[\mathcal{P}_\perp^2 + e^\phi \pi e^\phi \pi + e^{2\phi} \left(X - \frac{1}{4} \right) \right], \quad (18)$$

$$X \equiv (l^i_j)^2 + (p^+ \eta^2 - 2)^2 + 4p^+ \eta_i l^i_j \eta^j. \quad (19)$$

The analogous expressions for the string Lagrangian and Hamiltonian written in terms of 6 Cartesian coordinates Y^M and the associated momentum \mathcal{P}_M , i.e. corresponding to the Lagrangian (5) are

$$\mathcal{L} = \mathcal{P}_\perp \dot{x}_\perp + \mathcal{P}_M \dot{Y}^M + \frac{i}{2} p^+ (\theta^i \dot{\theta}_i + \eta^i \dot{\eta}_i + \theta_i \dot{\theta}^i + \eta_i \dot{\eta}^i) - \mathcal{H}, \quad (20)$$

$$\begin{aligned} \mathcal{H} = & \frac{1}{2p^+} \left(\mathcal{P}_\perp^2 + Y^4 \dot{x}_\perp^2 + Y^4 \mathcal{P}_M \mathcal{P}_M + \dot{Y}^M \dot{Y}^M \right. \\ & \left. + Y^2 [p^{+2} (\eta^2)^2 + 2ip^+ \eta \rho^{MN} \eta Y_M \mathcal{P}_N] \right) - |Y| Y^M \left[\eta \rho^M (\dot{\theta} - i\sqrt{2} |Y| \eta \dot{x}) + h.c. \right], \end{aligned} \quad (21)$$

$$p^+ \dot{x}^- + \mathcal{P}_\perp \dot{x}_\perp + \mathcal{P}_M \dot{Y}^M + \frac{i}{2} p^+ (\theta^i \dot{\theta}_i + \eta^i \dot{\eta}_i + \theta_i \dot{\theta}^i + \eta_i \dot{\eta}^i) = 0. \quad (22)$$

The fact that in the particle theory limit the string Hamiltonian (17) or (21) reduces ¹⁵ to the superparticle Hamiltonian ²⁵ implies that the “massless” (zero-mode)

^gWe explicitly introduce momenta only for the bosons. The odd (fermionic) part of the phase space may be viewed as represented by θ^i, η^i considered as fermionic coordinates and θ_i, η_i considered as fermionic momenta.

spectrum of the superstring coincides indeed with the spectrum of type IIB supergravity compactified on S^5 . The vertex operators for the supergravity states are then obtained by solving the linearized supergravity equations (or 1-st quantized superparticle state equations) in $AdS_5 \times S^5$ background.

To restore the dependence on R and α' one needs to make the replacements: $e^\phi \rightarrow e^{\phi/R}$, $u^M \rightarrow Ru^M$, $\mathcal{P} \rightarrow T^{-1/2}\mathcal{P}$, $x' \rightarrow T^{1/2}x'$, etc., where $T = (2\pi\alpha')^{-1}$. It may be interesting to study the limiting cases of the parameters, like $\alpha'/R^2 \rightarrow \infty$ for fixed momenta, etc. $R \rightarrow \infty$ gives of course the standard flat space light-cone superstring Hamiltonian (in the specific parametrization of fermions we are using, see section 2).

For generic values of parameters the Hamiltonian (17),(21) looks quite non-linear. Further progress may depend on a possibility of making a transformation to some new variables which may allow one to solve for the string theory spectrum. Such transformation should involve fermions in an essential way. The phase space formulation seems a good starting point for searching for such a transformation since the bosonic momenta are already nontrivially dependent on the fermions. One potentially interesting idea is to try to introduce twistor-like variables (see ³¹ for some previous discussions of twistors in AdS space).

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